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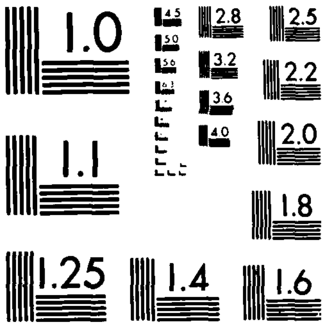

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THE EFFECT OF COORDINATE SYSTEM ROTATION ON SPHERICAL HARMONIC EXPANSIONS: A NUMERICAL METHOD

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A numerical method is presented for effecting the transformation of a spherical harmonic expansion under an arbitrary rotation of the coordinate system. Included are results of numerical tests with expansions to degree and order 180. These results demonstrate the high accuracy of the algorithm.

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TABLE OF CONTENTS

LIST OF FIGURES

1. INTRODUCTION

This report presents a numerical method for computing the effects of an arbitrary rotation of the coordinate system on a spherical harmonic expansion. The analysis was motivated by a study of the approximations involved in the use of the Fourier In-Flight Transfer Function for the evaluation of high-frequency gravity missile impact errors (Ref. 6).

The method has a wide range of applications. In Ref. 6 it was used to minimize latitude distortion on a planar grid representing surface gravity in the neighborhood of the launch point by redefining the coordinate system to make the (false) equator pass through the launch point. Similar applications exist to data reduction and model validation where reference/model gravity, usually described by a spherical harmonic expansion, has to be evaluated on a local grid in the region where data exists. Other examples of possible applications are:

- Generation of polar charts and maps where any given point is the pole of the expansion
- Direct simulation and gravity model generation for false-pole implementations of inertial navigation and guidance systems
- Comparison and correlation studies of different geophysical global models (Ref. 3)
- Theoretical analyses of isotropic and stationary properties of the gravity field.

The problem of determining the transformation of a spherical harmonic expansion under arbitrary rotations of the coordinate system has received considerable attention in the past. Equivalent formulas have been independently derived by several investigators. The earliest reference is to A. Schmidt and dates back to 1899 (Ref. 13). Other works often quoted in the literature are due to Herglotz (Ref. 4), Wigner (Ref. 15) and Jeffreys (Ref. 9). More recently, Kleusberg (Ref. 12) has derived an approximation valid for small rotations. Applications to Geodesy and the study of satellite orbits have been given by Kaula (Ref. 10), Cook (Ref. 2), Izsak (Ref. 8), Balmino and Borderies (Ref. 1) and Giacaglia (Ref. 5).

Because of the complexity of the general transformation formula and because of numerical instability in the propagation of the tranformation coefficients, the formula has only been used for low degree (<10) expansions. The numerical method presented in this report has been tested and yields accurate results with expansions to degree 180. The method is based on a decomposition of an arbitrary rotation of the coordinate system, R, into five elementary rotations in the form

$$R = A_\gamma \; B^{-1} \; A_\beta \; B \; A_\alpha \tag{1-1}$$

where A_α, A_β, and A_γ represent polar axis rotations and B and B^{-1} are 90-degree rotations about a fixed equatorial axis whose effect on the spherical harmonics is described by a set of precomputed weighting coefficients.

This report is organized as follows: Section 2 presents the derivation of Eq. 1-1 and discusses the effect of each of the rotations A_α, A_β, A_γ, B and B^{-1} on the spherical harmonic coefficients. In Section 3 an algorithm is given for

the evaluation of the weighting coefficients associated with rotations B and B^{-1}. Section 4 presents some numerical results obtained with expansions to degree and order 180. Finally, a summary of the contents of this report is given in Section 5.

2. ALGORITHM FOR ROTATION OF SPHERICAL HARMONICS

In this section, a discussion of two different conventional definitions of spherical harmonics is given first. Simple relations are given for transforming coefficients from one convention to the other. This discussion is followed by a presentation of the general formula for rotation of spherical harmonics. Finally, the algorithm for practical implementation of this transformation is introduced.

Perhaps the most widely accepted definition of normalized surface spherical harmonics is that of Ref. 7:

$$R_{\ell 0}(\theta,\lambda) = (2\ell+1)^{1/2} P_{\ell 0}(\cos\theta)$$

for $\ell \geq 0$, and

$$\begin{Bmatrix} R_{\ell m}(\theta,\lambda) \\ S_{\ell m}(\theta,\lambda) \end{Bmatrix} = \left[2(2\ell+1)\frac{(\ell-m)!}{(\ell+m)!}\right]^{1/2} P_{\ell m}(\cos\theta) \begin{Bmatrix} \cos m\lambda \\ \sin m\lambda \end{Bmatrix} \tag{2-1}$$

for $1 \leq m \leq \ell$, where $P_{\ell m}$ is the associated Legendre function of degree ℓ and order m given by

$$P_{\ell m}(x) = \frac{(-1)^{\ell+m}}{2^{\ell}2!}\frac{(\ell+m)!}{(\ell-m)!}(1-x^2)^{-m/2}\frac{d^{\ell-m}}{dx^{\ell-m}}(1-x^2)^{\ell} \tag{2-2}$$

with θ and λ as in Fig. 2-1. However, the equations and results in this report are considerably simplified when complex spherical harmonics are used. The normalized complex surface spherical harmonics are defined as (Ref. 4)

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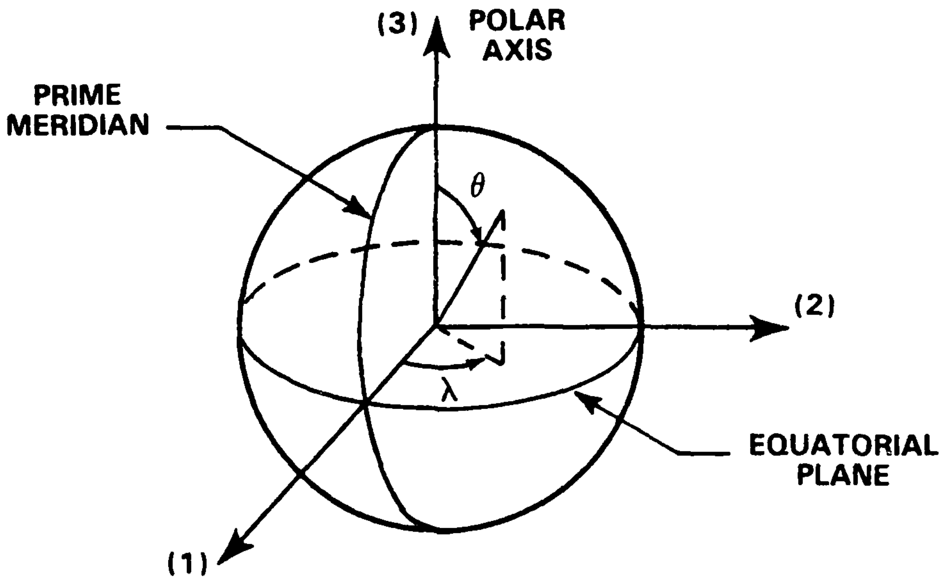

Figure 2-1 Spherical Coordinates

$$Y_{\ell}^{m}(\theta,\lambda) = \left[(2\ell+1) \frac{(\ell-m)!}{(\ell+m)!} \right]^{1/2} P_{\ell m}(\cos\theta)\, e^{im\lambda} \qquad (2\text{-}3)$$

where $i = \sqrt{-1}$.

To simplify the terminology, in the sequel the functions defined by Eq. 2-1 are referred as "real harmonics" and those given by Eq. 2-3 as "complex harmonics" or simply as "harmonics" when the meaning is clear from the context.

For fixed degree ℓ and nonzero order m, there are two real harmonics: $R_{\ell m}$ and $S_{\ell m}$. This is also the case for the complex harmonics. The functions Y_{ℓ}^{m} and Y_{ℓ}^{-m} correspond to order $|m|$. In other words, the definitions in Eqs. 2-2 and 2-3 also apply to negative m as long as $|m| \leq \ell$. It can be shown that

$$Y_{\ell}^{-m}(\theta,\lambda) = (-1)^m \, \overline{Y}_{\ell}^{m}(\theta,\lambda) \tag{2-4}$$

where the superimposed bar denotes complex conjugation.

The expansion of a function $h(\theta,\lambda)$ in a series of complex harmonics is of the form

$$h(\theta,\lambda) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} C_{\ell}^{m} \, Y_{\ell}^{m}(\theta,\lambda) \tag{2-5}$$

where the coefficients C_{ℓ}^{m} are complex numbers given by

$$C_{\ell}^{m} = \frac{1}{4\pi} \int_{\sigma} h(\theta,\lambda) \, \overline{Y}_{\ell}^{m}(\theta,\lambda) \, d\sigma \tag{2-6}$$

Since all functions $h(\theta,\lambda)$ considered in this report are real, it follows from Eqs. 2-4 and 2-6 that the coefficients C_{ℓ}^{m} must satisfy

$$C_{\ell}^{-m} = (-1)^m \, \overline{C}_{\ell}^{m} \tag{2-7}$$

Now, suppose that the expansion of $h(\theta,\lambda)$ in real harmonics is

$$h(\theta,\lambda) = \sum_{\ell=0}^{\infty} \sum_{m=0}^{\ell} a_{\ell m} \, R_{\ell m}(\theta,\lambda) + \sum_{\ell=1}^{\infty} \sum_{m=1}^{\ell} b_{\ell m} \, S_{\ell m}(\theta,\lambda) \tag{2-8}$$

Using Eqs. 2-1, 2-3, 2-4, and 2-5 it can be shown that

$$C_\ell^0 = a_{\ell 0} \qquad \ell \geq 0 \tag{2-9}$$

and

$$C_\ell^m = (a_{\ell m} - ib_{\ell m})/\sqrt{2} \qquad 1 \leq m \leq \ell \tag{2-10}$$

These relations, together with Eq. 2-4 permit direct translation of expansions in real harmonics into expansions in complex harmonics and vice versa.

Next, consider a rotation, R, of the coordinate system through the Euler angles α, β, γ defined as in Fig. 2-2. It is shown in Ref. 4 that a harmonic $Y_\ell^m(\theta,\lambda)$ is mapped into a linear combination of harmonics of the same degree in the form

$$Y_\ell^m(\theta,\lambda) = \sum_{j=-\ell}^{\ell} Q_\ell^{m,j} \; Y_\ell^j(\theta',\lambda') \tag{2-11}$$

where θ' and λ' are colatitude and longitude in the rotated coordinate system and where the weights, $Q_\ell^{m,j}$, depend on the Euler angles. Combining the formulas for the weights given in Ref. 4 with the results of Szego (Ref. 14) on extensions of Jacobi Polynomials, $P_k^{(\nu_1,\nu_2)}$, to all possible values of the superindices, it can be shown that

$$Q_\ell^{m,j} = (-1)^{\ell-m} \frac{\binom{2\ell}{\ell+m}^{1/2}}{\binom{2\ell}{\ell+j}^{1/2}} \, i^{j-m} \, e^{i(m\alpha+j\gamma)}$$

$$\times \cos^{m+j}(\beta/2) \, \sin^{j-m}(\beta/2) \, P_{\ell-j}^{(m+j,j-m)}(-\cos\beta) \tag{2-12}$$

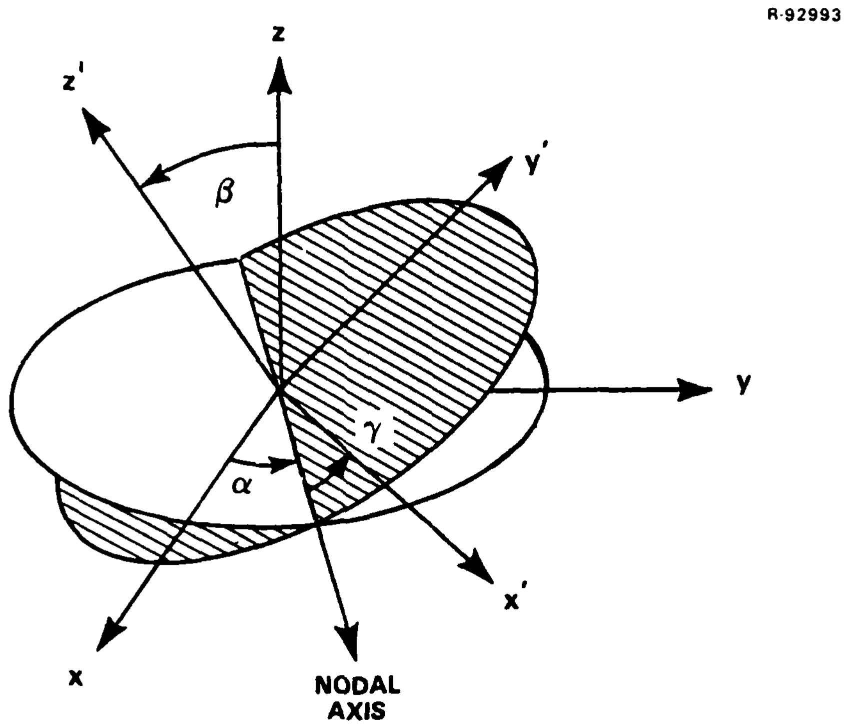

Figure 2-2 Euler Angles. The angles α, β, γ are successive rotations about the Z axis, the nodal axis, and the Z' axis, respectively.

Using Eq. 2-11, the expansion in the original coordinate system, Eq. 2-5, can be transformed into an expansion in the primed system. The result is

$$\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} C_{\ell}^{m} Y_{\ell}^{m}(\theta,\lambda) = \sum_{\ell=0}^{\infty} \sum_{j=-\ell}^{\ell} \left[\sum_{m=-\ell}^{\ell} Q_{\ell}^{m,j} C_{\ell}^{m} \right] Y_{\ell}^{j}(\theta',\lambda') \tag{2-13}$$

The expression in brackets is readily recognized as the coefficient of degree ℓ and order j, D_{ℓ}^{j}, in the new coordinate system. Thus,

$$D_{\ell}^{j} = \sum_{m=-\ell}^{\ell} Q_{\ell}^{m,j} C_{\ell}^{m} \tag{2-14}$$

If the above relation is viewed as a computational equation, the transformation weights, $Q_{\ell}^{m,j}$, have to be evaluated each time a new choice of Euler angles is used. The dependence on the angles α and γ presents no difficulty. However, recursions on $Q_{\ell}^{m,j}$ which are valid for a range of values of β rapidly become numerically unstable.

The alternative suggested in this report is to decompose the general rotation of Fig. 2-2 into a collection of five successive rotations each of which has transformation weights which are either dependent upon an angle but easily computed or are fixed and precomputed. The decomposition and the transformation weights associated with the component rotations are discussed below.

Next, it is shown that the general rotation of Fig. 2-2 can be decomposed as

$$R = A_{\gamma}\ B^{-1}\ A_{\beta}\ B\ A_{\alpha} \tag{2-15}$$

where A_{ϕ} represents a rotation by an angle ϕ about the axis of the third coordinate (the polar axis), and B is a rotation of $\pi/2$ about the axis of the second coordinate. (Figure 2-1 defines the ordering of the axes.)

The appearance of the first and last terms on the right side of Eq. 2-15 is clear from Fig. 2-2. They correspond, respectively, to the rotations about the transformed and the original polar axes in Fig. 2-2. Consider now the factor $B^{-1}\ A_{\beta}\ B$ in Eq. 2-15. The matrix representations of A_{β} and B are

$$A_\beta = \begin{pmatrix} \cos\beta & \sin\beta & 0 \\ -\sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2-16}$$

and

$$B = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{2-17}$$

Therefore, the product $B^{-1}A_\beta B$ is

$$B^{-1}A_\beta B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & \sin\beta \\ 0 & -\sin\beta & \cos\beta \end{pmatrix} \tag{2-18}$$

which is precisely the matrix representation of the intermediate rotation in Fig. 2-2. This shows the validity of the decomposition formula (Eq. 2-15).

Thus, the algorithm for transformation of spherical harmonic expansions consists of successively computing the effect of rotations A_α, B, A_β, B^{-1} and A_γ on the harmonic coefficients. The transformation weights associated with each of these rotations are discussed below. Note that only those coefficients with nonnegative upper index need to be computed at each stage because the function represented by the expansion is real and, therefore, each of the rotations will yield coefficients which satisfy Eq. 2-7.

The transformation weights of a polar rotation, A_ϕ, are a simple phase change applied separately to each of the harmonic coefficients. If C_ℓ^m and D_ℓ^m are the original and transformed coefficients then

$$D_\ell^m = e^{im\phi} C_\ell^m \qquad (2\text{-}19)$$

This can be demonstrated directly from the definition of complex harmonics, Eq. 2-3, by redefining the longitude reference as $\lambda' = \lambda - \phi$.

The transformation weights of rotation B are given by Eq. 2-12 with $\alpha = \pi/2$, $\beta = \pi/2$ and $\gamma = -\pi/2$. The expression for these weights, which are denoted by $T_\ell^{m,j}$ in the sequel, reduces to

$$T_\ell^{m,j} = (-1)^{\ell-m}\, 2^{-j} \frac{\binom{2\ell}{\ell+m}^{1/2}}{\binom{2\ell}{\ell+j}^{1/2}} P_{\ell-j}^{(m+j,j-m)}(0) \qquad (2\text{-}20)$$

Thus, all $T_\ell^{m,j}$ are real. These weights also possess useful symmetry properties which are given below.

Using the results of Ref. 14 it can be shown that the Jacobi polynomials satisfy

$$P_{\ell-j}^{(m+j,j-m)}(0) = 2^{j-m} \frac{\binom{2\ell}{\ell+j}}{\binom{2\ell}{\ell+m}} P_{\ell-m}^{(m+j,m-j)}(0) \qquad (2\text{-}21)$$

It then follows from the last two equations that

$$T_{\ell}^{m,j} = (-1)^{\ell-m}\, 2^{-m} \frac{\binom{2\ell}{\ell+j}^{1/2}}{\binom{2\ell}{\ell+m}^{1/2}} P_{\ell-m}^{(m+j,m-j)}(0) \tag{2-22}$$

Hence,

$$T_{\ell}^{m,j} = (-1)^{j-m}\, T_{\ell}^{j,m} \tag{2-23}$$

which implies that the numerical evaluation of $T_{\ell}^{m,j}$ can be limited to $j \geq m$.

Similarly, it can be shown that

$$T_{\ell}^{-m,j} = (-1)^{\ell-j}\, T_{\ell}^{m,j} \tag{2-24}$$

This relation, and the fact that $T_{\ell}^{m,j}$ is real, reduce the computational burden of the effect of rotation B on the harmonic coefficients by a factor of four. As before, let C_{ℓ}^{m} and D_{ℓ}^{m} be the original and transformed coefficients. Then, from Eqs. 2-14, 2-23 and 2-24,

$$D_{\ell}^{j} = T_{\ell}^{0,j}\, C_{\ell}^{0} + \sum_{m=1}^{\ell} T_{\ell}^{m,j}\, [C_{\ell}^{m} + (-1)^{\ell-(j+m)}\, \bar{C}_{\ell}^{m}] \tag{2-25}$$

When compared with Eq. 2-14, the above relation shows a factor of two reduction in the number of multiplications because of the disappearance of negative indices in the summation. In addition, since the quantity in brackets is either real or purely imaginary, another factor of two is gained.

To conclude this section, consider rotation B^{-1}. The Euler angles are $\alpha = \pi/2$, $\beta = -\pi/2$, and $\gamma = -\pi/2$. It then follows from Eq. 2-12 that the transformation weights of B^{-1}, denoted by $W_\ell^{m,j}$, are given by

$$W_\ell^{m,j} = (-1)^{\ell-j}\, 2^{-j}\, \frac{\binom{2\ell}{\ell+m}^{1/2}}{\binom{2\ell}{\ell+j}^{1/2}}\, P_{\ell-j}^{(m+j,j-m)}(0) \qquad (2\text{-}26)$$

Comparing this expression with Eq. 2-22, it can be seen that the weights of B^{-1} are identical to the weights of B except for transposition of the upper indices; i.e.,

$$W_\ell^{m,j} = T_\ell^{j,m} \qquad (2\text{-}27)$$

Therefore, a single table of weights is sufficient to describe the effect of both B and B^{-1} on the harmonic coefficients. In terms of $T_\ell^{m,j}$, the effect of B^{-1} is

$$D_\ell^j = T_\ell^{j,0}\, C_\ell^0 + \sum_{m=1}^{\ell} T_\ell^{j,m}\, [C_\ell^m + (-1)^{\ell-(j+m)}\, \bar{C}_\ell^m] \qquad (2\text{-}28)$$

3. EVALUATION OF THE WEIGHTS

The transformation weights, $T_\ell^{m,j}$, are evaluated through recursion on the three indices. All recursion relations used in the algorithm given below were derived from formulas in Ref. 14 relating contiguous Jacobi polynomials. The slowest varying index in the algorithm is ℓ. Thus, all weights $T_\ell^{m,j}$ for fixed degree ℓ can be stored as a single record. Later, when the table of weights is used to rotate a spherical harmonic expansion, these weights can also be read as a single record and all the coefficients of degree ℓ can be processed at once.

The weights of degree 0 and 1 serve as starting values for the computation. They are given in Table 3-1 which also lists all the weights through degree five. Only the weights $T_\ell^{m,j}$ for $0 \leq m \leq j \leq \ell$ need to be computed because of the symmetry of the weights (Eqs. 2-23 and 2-24) and because the algorithm for rotation of spherical harmonic coefficients only has to evaluate the transformed coefficients, D_ℓ^j, for $j \geq 0$.

The following notation is used: Int(x) denotes the largest integer less than or equal to x and μ_ℓ^k is defined as

$$\mu_\ell^k = [(\ell+k+1)(\ell-k)]^{1/2} \qquad (k=0,\ldots,\ell) \qquad (3\text{-}1)$$

Next, the algorithm is presented. A discussion of its various steps is given at the end of this section.

TABLE 3-1

WEIGHTS $T_{\ell}^{m,j}$ FOR $\ell \leq 5$

ℓ	m	j=0	j=1	j=2	j=3	j=4	j=5
0	0	1					
1	0	0	$-\sqrt{2}/2$				
1	1	$\sqrt{2}/2$	1/2				
2	0	-1/2	0	$\sqrt{6}/4$			
2	1	0	-1/2	-1/2			
2	2	$\sqrt{6}/4$	1/2	1/4			
3	0	0	$\sqrt{3}/4$	0	$-\sqrt{5}/4$		
3	1	$-\sqrt{3}/4$	-1/8	$\sqrt{10}/8$	$\sqrt{15}/8$		
3	2	0	$-\sqrt{10}/8$	-1/2	$-\sqrt{6}/8$		
3	3	$\sqrt{5}/4$	$\sqrt{15}/8$	$\sqrt{6}/8$	1/8		
4	0	3/8	0	$-\sqrt{10}/8$	0	$\sqrt{70}/16$	
4	1	0	3/8	$\sqrt{2}/8$	$-\sqrt{7}/8$	$-\sqrt{14}/8$	
4	2	$-\sqrt{10}/8$	$-\sqrt{2}/8$	1/4	$\sqrt{14}/8$	$\sqrt{7}/8$	
4	3	0	$-\sqrt{7}/8$	$-\sqrt{14}/8$	-3/8	$-\sqrt{2}/8$	
4	4	$\sqrt{70}/16$	$\sqrt{14}/8$	$\sqrt{7}/8$	$\sqrt{2}/8$	1/16	
5	0	0	$-\sqrt{30}/16$	0	$\sqrt{35}/16$	0	$-3\sqrt{7}/16$
5	1	$\sqrt{30}/16$	1/16	$-\sqrt{7}/8$	$-\sqrt{42}/32$	$\sqrt{21}/16$	$\sqrt{210}/32$
5	2	0	$\sqrt{7}/8$	1/4	$-\sqrt{6}/16$	$-\sqrt{3}/4$	$-\sqrt{30}/16$
5	3	$-\sqrt{35}/16$	$-\sqrt{42}/32$	$\sqrt{6}/16$	13/32	$9\sqrt{2}/32$	$3\sqrt{5}/32$
5	4	0	$-\sqrt{21}/16$	$-\sqrt{3}/4$	$-9\sqrt{2}/32$	-1/4	$-\sqrt{10}/32$
5	5	$3\sqrt{7}/16$	$\sqrt{210}/32$	$\sqrt{30}/16$	$3\sqrt{5}/32$	$\sqrt{10}/32$	1/32

Algorithm for Evaluation of the Weights $T_\ell^{m,j}$ $(0 \le m \le j \le \ell \le L)$

1. [Initialization]

   Let $\ell=2$, $T_0^{0,0}=1$, $T_1^{0,0}=0$, $T_1^{0,1}=-\sqrt{2}/2$ and $T_1^{1,1}= 1/2$

   In addition set $a \leftarrow T_1^{0,0}$, $b \leftarrow T_1^{0,1}$, $c \leftarrow T_1^{0,1}$ and $d \leftarrow -T_1^{0,1}$

2. [Determine $T_\ell^{0,0}$ and $T_\ell^{0,1}$]

   If ℓ is even then set $a \leftarrow -\frac{\ell-1}{\ell}a$ and put $T_\ell^{0,0} = a$, $T_\ell^{0,1}=0$

   If ℓ is odd then set $b \leftarrow -\left[\frac{\ell(\ell-2)}{(\ell+1)(\ell-1)}\right]^{1/2} b$ and put $T_\ell^{0,0}=0$, $T_\ell^{0,1}=b$

3. [Determine $T_\ell^{0,j}$ for $j=2, \ldots, \ell-1$]

   Use the following formula recursively

$$T_\ell^{0,j} = -\frac{\mu_\ell^{j-2}}{\mu_\ell^{j-1}} T_\ell^{0,j-2} \tag{3-2}$$

4. [Determine $T_\ell^{0,\ell}$ and $T_\ell^{1,\ell-1}$]

   Set

$$c \leftarrow -\left[\frac{2\ell-1}{2\ell}\right]^{1/2} c \quad \text{and} \quad d \leftarrow -\left[\frac{2\ell-1}{2\ell+2}\right]^{1/2} d$$

   and put $T_\ell^{0,\ell} = c$, $T_\ell^{1,\ell-1} = d$

5. [Determine $T_{\ell}^{m,\ell}$ for $m=1, \ldots, \ell$]

Use the following formula recursively

$$T_{\ell}^{m,\ell} = - \left[\frac{\ell-m+1}{\ell+m}\right]^{1/2} T_{\ell}^{m-1,\ell} \tag{3-3}$$

6. [Determine $T_{\ell}^{m,\ell-1}$ for $m=2, \ldots, \ell-1$]

Use the following formula recursively

$$T_{\ell}^{m,\ell-1} = - \frac{m}{m-1} \left[\frac{\ell-m+1}{\ell+m}\right]^{1/2} T_{\ell}^{m-1,\ell-1} \tag{3-4}$$

7. [Determine $T_{\ell}^{m,j}$ for $2 \leq m \leq j \leq \ell-2$]

7.1 Set $k = \max\{0, \ell-2-\mathrm{Int}[2(\ell-1)^{1/2}]\}$

7.2 [Case $j-m \geq k$]

For each $n = \ell-3, \ell-4, \ldots, k$ let $j = \ell-2, \ell-3, \ldots, n+1$ and set $m \leftarrow j-n$. Compute

$$T_{\ell}^{m,j} = - \frac{m}{j+1} \frac{\mu_{\ell}^{m}}{\mu_{\ell}^{j}} T_{\ell}^{m+1,j+1} - \frac{m}{j+1} \frac{\mu_{\ell}^{m-1}}{\mu_{\ell}^{j}} T_{\ell}^{m-1,j+1} + \\ - \frac{\mu_{\ell}^{j+1}}{\mu_{\ell}^{j}} T_{\ell}^{m,j+2} \tag{3-5}$$

7.3 [Case $j-m<k$]

For each $n=k-1, k-2, \ldots, 0$ set $q \leftarrow \text{Int}\{[n+(2\ell^2-n^2)^{1/2}]/2\}$ and perform the following two steps in the order given

7.3.1 For $j=q+1, q+2, \ldots, \ell-2$ set $m \leftarrow j-n$ and compute

$$T_\ell^{m,j} = \frac{j-m+1}{j+m-1} \frac{\mu_\ell^{m-2}}{\mu_\ell^{m-1}} T_\ell^{m-2,j} - \frac{2j}{j+m-1} \frac{\mu_\ell^{j}}{\mu_\ell^{m-1}} T_\ell^{m-1,j+1} \tag{3-6}$$

7.3.2 For $j=q, q-1, \ldots, n+1$ set $m \leftarrow j-n$ and compute $T_\ell^{m,j}$ using Eq. 3-5

8. [Update degree]

All weights of degree ℓ have been computed. If $\ell<L$ then set $\ell \leftarrow \ell+1$ and go to Step 2.

Figure 3-1 presents a graphical interpretation of the operation of the algorithm. The numbers shown in parentheses next to the arrows are the steps of the algorithm in which the weights along the lines parallel to the arrows are computed. The direction of the arrows indicates the progression in which the weights are evaluated.

Four quantities (denoted by a, b, c, and d in the algorithm) are needed to propagate the weights from degree $\ell-1$ to degree ℓ. They are initialized in Step 1 and updated in Steps 2 and 4 where they are also used to evaluate four weights of degree ℓ: $T_\ell^{0,0}$, $T_\ell^{0,1}$, $T_\ell^{0,\ell}$ and $T_\ell^{1,\ell-1}$. These weights, indicated by circles in Fig. 3-1, serve as "seeds" for the computation of the remaining weights of degree ℓ.

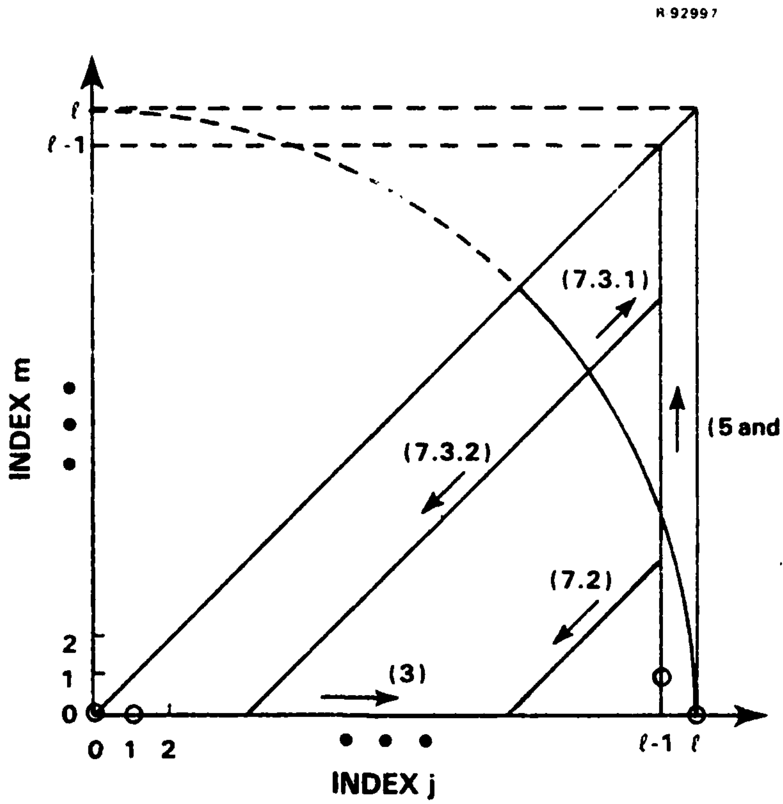

Figure 3-1 Computation of the Weights of Degree ℓ

The main recursion formula is Eq. 3-5. It permits the propagation of the weights along lines parallel to the diagonal (Fig. 3-1). Its application, however, requires prior evaluation of the weights along the boundary lines $m=0$, $j=\ell-1$, and $j=\ell$. This is achieved in Steps 3, 5 and 6 of the algorithm.

The main recursion formula, Eq. 3-5, is numerically unstable outside the circle $m^2+j^2 \leq \ell^2$ (See Fig. 3-1). This is the reason for dividing Step 7 into two separate cases. If the weights to be computed along the subdiagonal $j-m=n$(=constant) all lie inside this circle then Eq. 3-5 is used. If they do not, then another recursion formula, Eq. 3-6, is necessary. The critical value of n is computed in Step 7.1 and is denoted by k in the algorithm. Thus, in the region $j-m \geq k$ the main recursion formula is used (Step 7.2) while for $j-m<k$, Eq. 3-6 is

used outside the circle (Step 7.3.1) and the main recursion formula is used inside the circle (Step 7.3.2).

Double precision (53 bit mantissa) was used in an IBM 4341 computer to generate all weights to degree 180. CPU time was approximately 3 minutes. The accuracy of the computed weights was later verified using recursion formulas independent of those used in the algorithm described above. These formulas are

$$T_\ell^{1,j} = \begin{cases} -\dfrac{j}{\mu_\ell^0} T_\ell^{0,j} & \text{if } \ell-j \text{ is even} \\ \\ -\dfrac{\mu_\ell^j}{\mu_\ell^0} T_\ell^{0,j+1} & \text{if } \ell-j \text{ is odd} \end{cases} \tag{3-7}$$

and

$$T_\ell^{m,j} = 2\,\frac{j+m+2}{j+m+3}\,\frac{(\ell-m-1)(\ell-j-1)+(\ell-m-j-1)(j+m+3)}{\mu_\ell^j\,\mu_\ell^m}\,T_\ell^{m+1,j+1}$$

$$-\frac{j+m+1}{j+m+3}\,\frac{\mu_\ell^{j+1}\,\mu_\ell^{m+1}}{\mu_\ell^j\mu_\ell^m}\,T_\ell^{m+2,j+2} \tag{3-8}$$

Equation 3-7 permits the computation of all weights on the line m=1 in terms of those for m=0. Note that $T_\ell^{1,j}$ is the last weight evaluated along each subdiagonal using the main recursion formula in Steps 7.2 and 7.3.2 (See Fig. 3-1). Therefore, a comparison of the weights $T_\ell^{1,j}$ generated by the algorithm and those computed on the basis of Eq. 3-7 provides a measure of the numerical behavior of the main recursion formula.

On the other hand, Eq. 3-8 gives $T_{\ell}^{m,j}$ in terms of two neighbors on the same subdiagonal. This formula is not numerically stable. It was used only as a check on numerical closure by repeatedly substituting weights generated by the algorithm for $T_{\ell}^{m+1,j+1}$ and $T_{\ell}^{m+2,j+2}$ on the right hand side of Eq. 3-8 and comparing the result with the value of $T_{\ell}^{m,j}$ produced by the algorithm.

The tests showed that the accuracy of the computed weights degraded with decreasing absolute magnitude of the weights themselves. No comparison showed less than six (decimal) significant figures of agreement. Since the order of magnitude of the weights ranges over five decades and since the smaller the magnitude of the weight, the smaller its effect in the evaluation of harmonic coefficients (see Eq. 2-25), the results obtained with the algorithm were considered more than adequate. Further proof of the validity of the computation of the weights is presented in the next section.

4. NUMERICAL RESULTS

The algorithm for rotation of spherical harmonics was tested using three synthetic expansions, all to degree 180. In each expansion, the coefficients were zero-mean normal variates chosen according to Kaula's rule for the degree variances of the geopotential (Ref. 10). The population variance for each of the coefficients of degree ℓ in the expansions in real spherical harmonics was

$$\sigma_\ell^2 = 10^{-10}/\ell^4 \qquad \ell > 0 \tag{4-1}$$

The series were first rotated through random Euler angles and then transformed back to the original coordinate system. The resulting and the original coefficients were compared and differences were determined. Double precision was used throughout the computation.

Figures 4-1 and 4-2 summarize the results. In Fig. 4-1 a plot is given of the RMS error as a fraction of the expected value of the magnitude of the coefficients. The plot was obtained by combining the results of the three test cases. Thus, in evaluating the RMS error in the coefficients of degree ℓ, $6\ell+3$ differences were considered. The resulting value was normalized by the expected magnitude of the coefficients given by

$$k_\ell = 2\sigma_\ell/\sqrt{2\pi} \tag{4-2}$$

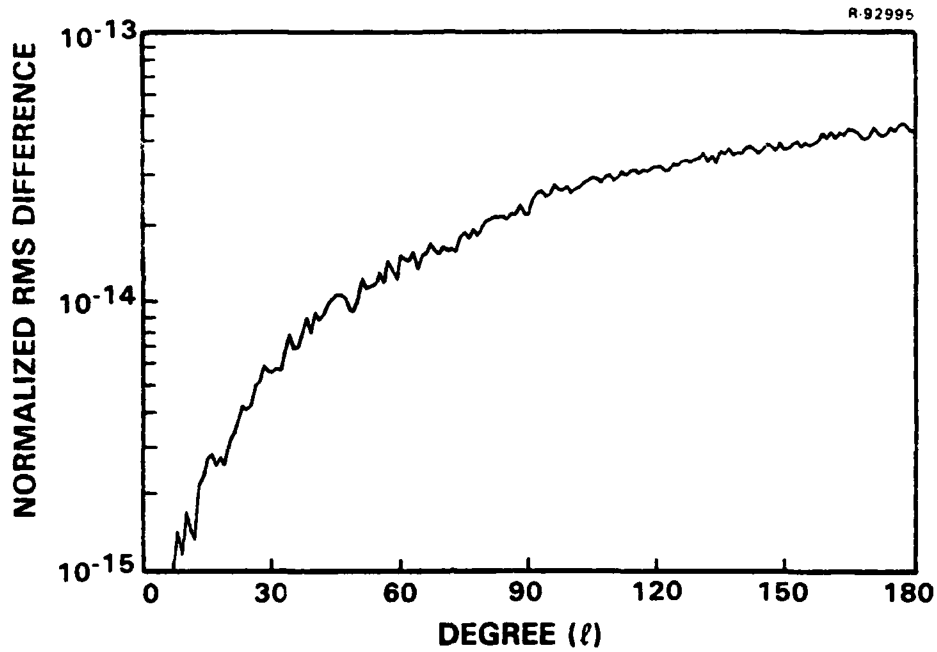

Figure 4-1 RMS Difference as a Fraction of Average Coefficient Magnitude

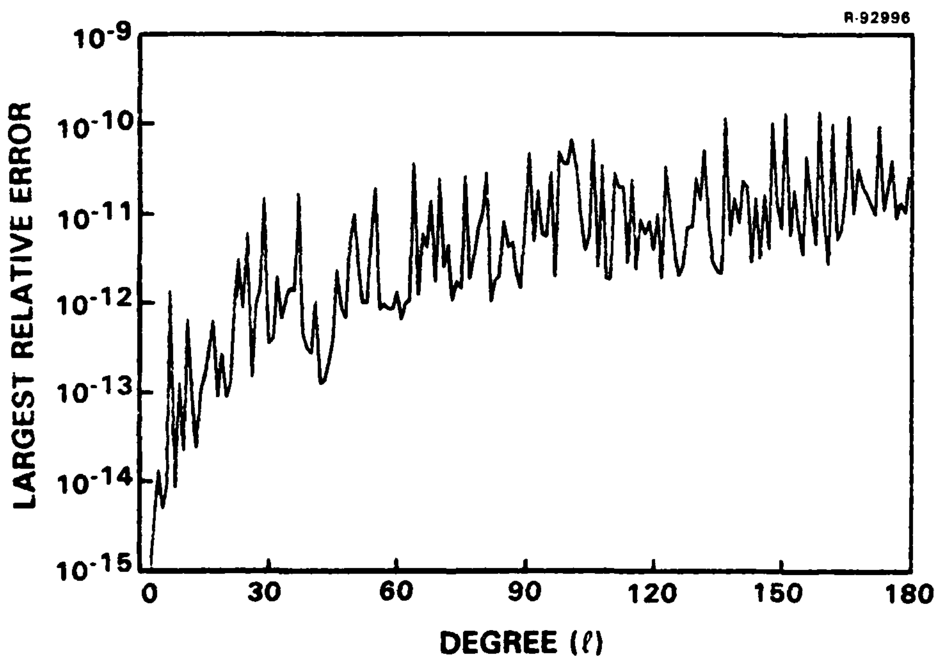

Figure 4-2 Largest Relative Errors Found in the Numerical Tests

with σ_ℓ as in Eq. 4-1. The purpose of this normalization was to compensate for the decreasing magnitude of the coefficients in order to provide a measure of the number of exact significant figures which can be expected from the application of the algorithm. Thus, Fig. 4-1, indicates that up to degree 180 the standard deviation of the errors is always more than thirteen orders of magnitude smaller than the average size of the coefficients.

Figure 4-2, on the other hand, is a plot of worst-case results. It shows the largest relative error found in all three test cases as a function of degree. In other words, for given degree ℓ, the largest ratio between the absolute values of the error and the true (original) coefficient in all test cases was plotted in Fig. 4-2. It was observed that the largest relative errors tend to occur in those coefficients whose magnitude is small compared with other coefficients of the same degree. This is because, in general, all of the coefficients of degree ℓ enter in the computation of any single coefficient of the same degree in the rotated expansion. Therefore, the absolute errors have the same order of magnitude for all computed coefficients but they represent a larger percent in the smaller coefficients. As Fig. 4-2 indicates, in no case did the computed coefficients have less than nine significant figures of agreement with the original coefficients.

The number of operations in the algorithm is $O(N^3)$ where N is the highest degree in the expansion. In an IBM 4341 computer, CPU time for a single (one-way) rotation was slightly more than two minutes for an expansion to degree 180 and less than three seconds for an expansion to degree 36.

5. SUMMARY CONCLUSIONS AND RECOMMENDATIONS

An algorithm has been given for accurately computing the effect of an arbitrary rotation of the coordinate system on a spherical harmonic expansion. The method consists of decomposing the rotation, R, into 5 successive component rotations in the form

$$R = A_\gamma \; B^{-1} \; A_\beta \; B \; A_\alpha \qquad (5\text{-}1)$$

where α, β and γ are the Euler angles of R.

The effect of A_α, A_β and A_γ on the harmonic coefficients is easily computed since they represent rotations about the polar axis. On the other hand, matrices B and B^{-1} correspond to a 90 degree rotation about an equatorial axis and its inverse. Their effect on the harmonic coefficients was shown to be described by a set of weights which is fixed and independent of rotation R. A stable algorithm was given for the evaluation of the weights.

Numerical tests with synthetic expansions to degree 180 were performed. These tests verified the methodology and demonstrated the high accuracy of the algorithm.

It is recommended that the algorithm be implemented in the Weapons Support System (WSS) as a general tool for the analysis and use of spherical harmonic geopotential models. Specific and immediate applications include data reduction, model validation, charting and false-pole gravity model generation.

REFERENCES

1. Balmino, G., and Borderies, N., "Gravitational Potential of Solid Bodies in the Solar System," Celestial Mechanics, Vol. 17, 1978, pp. 113-119.

2. Cook, G.E., "Perturbations of Satellite Orbits by Tesseral Harmonics in the Earth's Gravitional Potential," Planet. Space Sci., Vol. 11, 1963, pp. 797-815.

3. Eckhardt, D.H., "Correlations Between Global Features of Terrestrial Fields," Air Force Geophysics Laboratory, Terrestrial Sciences Division, Hanscom, Mass., January 1983. Submitted to the Journal of the International Association of Mathematical Geology.

4. Erdélyi, A., et al., Higher Transcendental Functions, Vol. 2, McGraw-Hill, New York, 1953, pp. 253-261.

5. Giacaglia, G.E.O., "Transformations of Spherical Harmonics and Applications to Geodesy and Satellite Theory," Studia Geoph. et Geod., Vol. 24, 1980, pp. 1-11.

6. Goldstein, J.D., "Transfer Functions for In-Flight Gravity Impact Error Computation," TR-1946-11, The Analytic Sciences Corp., Reading, Mass., May 1983.

7. Heiskanen, W.A., and Moritz, H., Physical Geodesy, W.H. Freeman, San Francisco, 1967, pp. 28-32.

8. Izsak, I.G., "Tesseral Harmonics of the Geopotential and Corrections to Station Coordinates," J. Geophys. Res., Vol. 69, No. 12, 1964, pp. 2621-2630.

9. Jeffreys, B., "Transformation of Tesseral Harmonics under Rotation," Geophys. J., Vol. 10, 1963, pp. 141-145.

10. Kaula, W.M., "Analysis of Gravitational and Geometric Aspects of Geodetic Utilization of Satellites," Geophys. J., Vol. 5, 1961, pp. 104-133.

11. Kaula, W.M., Theory of Satellite Geodesy, Blaisdell Publ., Waltham, Mass., 1966.

REFERENCES (Continued)

12. Kleusberg, A., "The Similarity Transformation of the Gravitational Potential," International Symposium of Geodesy and Physics of the Earth, Karl-Marx-Stadt., 1980.

13. Schmidt, A., "Formeln zur Transformation der Kugelfunktionen bei Linearer Änderung des Koordinatensystems," Zeitschrift für Mathematic und Physik, Vol. 44, 1899, pp. 327-338.

14. Szegö, G., Orthogonal Polynomials, Ch. 4, American Mathematical Society Colloquium Publications, Providence, R.I., 1939.

15. Wigner, E.P., Group Theory and its Application to the Quantum Mechanics of Atomic Spectra, Ch. 15, Academic Press, 1959.

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